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Near-Optimal Separators in String Graphs

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Let G be a string graph (an intersection graph of continuous arcs in the plane) with m edges. Fox and Pach proved that G has a separator consisting of $O(m^{3/4}\sqrt{\log m})$ vertices, and they conjectured that the bound of $O(\sqrt{m})$ actually holds. We obtain separators with $O(\sqrt{m}\log m)$ vertices.

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Let $G = (V,E)$ be a graph with n vertices. A *separator* in G is a set $S \subseteq V$ of vertices such that there is a partition $V = V_1 \cup V_2 \cup S$ with $|V_1|, |V_2| \leqslant \frac{2}{3}n$ and no edges connecting V_1 to V_2. The graph G is a *string graph* if it is an intersection graph of curves in the plane, *i.e.*, if there is a system $(\gamma_v : v \in V)$ of curves (continuous arcs) such that $\gamma_u \cap \gamma_v \neq \emptyset$ if and only if $\{u,v\} \in E(G)$ or $u = v$.

Fox and Pach [4] proved that every string graph has a separator with $O(m^{3/4}\sqrt{\log m})$ vertices, where m is the number of edges of G.

We should mention that they actually proved the result for the weighted case, where each vertex $v \in$V has a positive real weight, and the size of the components of $G \setminus S$ is measured by the sum of vertex weights (while the size of S is still measured as the number of vertices). Our result can also be extended to the weighted case, either by deriving it from the unweighted case along the lines of [4], or by using appropriate vertex-weighted versions (available in the cited sources) of the tools used in the proof. However, for simplicity, we stick to the unweighted case in this note.

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Pach and Fox conjectured that string graphs actually have separators of size $O(\sqrt{m})$ (which, if true, would be asymptotically optimal in the worst case). Earlier, in [3], they proved some special cases of this conjecture, most notably, if every two curves γ_u, γ_v in the string representation intersect in at most k points, where k is a constant. As they kindly informed me in February 2013, they also have an (unpublished) proof of existence of separators of size $O(\sqrt{n})$ in string graphs with maximum degree bounded by a constant. Here we obtain the following result.

Theorem 1. *Every string graph G with $m \geqslant 2$ edges has a separator with $O(\sqrt{m}\log m)$ vertices.*

Clearly, we may assume that G is connected, and then the theorem immediately follows from Lemmas 2 and 3 below. Lemma 2 combines the considerations of [4] with those of [6] and adjusts them for vertex congestion instead of edge congestion. Lemma 3 replaces an approximate duality between sparsity of edge cuts and edge congestion due to Leighton and Rao [7] used in [6] with an approximate duality between sparsity of vertex cuts and vertex congestion, which is an immediate consequence of the results of Feige, Hajiaghayi and Lee [2].

Fox and Pach [5] obtained several interesting applications of Theorem 1. Here we mention yet another consequence.

Crossing number versus pair-crossing number. The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum possible number of edge crossings in a drawing of G in the plane, while the *pair-crossing number* $\mathrm{pcr}(G)$ is the minimum possible number of pairs of edges that cross in a drawing of G.

One of the most tantalizing questions in the theory of graph drawing is whether $\mathrm{cr}(G) = \mathrm{pcr}(G)$ for all graphs G [8], and in the absence of a solution, researchers have been trying to bound $\mathrm{cr}(G)$ from above by a function of $\mathrm{pcr}(G)$. The strongest result so far by Tóth [10] was $\mathrm{cr}(G) = O(p^{7/4}(\log p)^{3/2})$, where $p = \mathrm{pcr}(G)$. It is based on the Fox–Pach separator theorem for string graphs discussed above, and by replacing their bound by Theorem 1 in Tóth's proof, one obtains the improved estimate $\mathrm{cr}(G) = O(p^{3/2}\log^2 p)$.

Vertex congestion in string graphs. Let $\mathcal{P}$ denote the set of all paths in G, and for each pair $\{u,v\} \in \binom{V}{2}$ of vertices, let $\mathcal{P}_{uv} \subseteq \mathcal{P}$ be all paths from u to v. An *all-pair unit-demand multicommodity flow* in G is a mapping $\varphi : \mathcal{P} \to [0,1]$ such that $\sum_{P\in\mathcal{P}_{uv}} \varphi(P) = 1$ for every $\{u,v\} \in \binom{V}{2}$. The *congestion* $\mathrm{cong}(w)$ of a vertex $w \in V$ under φ is the total flow through w where, for conformity with [2], we count only half of the flow through a path P if w is one of the endpoints of P. That is,

$$\mathrm{cong}(w) = \sum_{P\in\mathcal{P}:w \text{ internal vertex of } P} \varphi(P) + \frac{1}{2}\sum_{P\in\mathcal{P}:w \text{ endpoint of } P} \varphi(P).$$

We define $\mathrm{vcong}(G) := \min_\varphi \max_{w\in V} \mathrm{cong}(w)$, where the minimum is over all all-pair unit-demand multicommodity flows.[1]

Lemma 2. *If G is a connected string graph, then* $\mathrm{vcong}(G) \geqslant cn^2/\sqrt{m}$ *(for a suitable constant $c > 0$).*

Proof. Let φ be a flow for which $\mathrm{vcong}(G)$ is attained, and let $(\gamma_v : v \in V)$ be a string representation of G. We construct a drawing of K_V, the complete graph on the vertex set V, as follows.

We draw each vertex $v \in V$ as a point $p_v \in \gamma_v$, in such a way that all the p_v are distinct.

For every edge $\{u,v\} \in \binom{V}{2}$ of the complete graph, we pick a path P_{uv} from $\mathcal{P}_{uv}$ at random, where each $P \in \mathcal{P}_{uv}$ is chosen with probability $\varphi(P)$, the choices being independent for different $\{u,v\}$. Let us enumerate the vertices along P_{uv} as $v_0 = u, v_1, v_2, \dots, v_k = v$. Then we draw the edge $\{u,v\}$ of K_V in the following manner: We start at p_u, follow γ_u until some (arbitrarily chosen) intersection with γ_{v_1}, then we follow γ_{v_1} until some intersection with γ_{v_2}, *etc.*, until we reach γ_v and p_v on it.

Let us estimate the expected number of pairs $\{\{u,v\},\{u',v'\}\}$ of edges of K_V that intersect in this drawing.

The drawings of $\{u,v\}$ and $\{u',v'\}$ may intersect only if there are vertices $w \in P_{uv}$ and $w' \in P_{u'v'}$ such that $\gamma_w \cap \gamma_{w'} \neq \emptyset$, *i.e.*, $\{w,w'\} \in E(G)$ or $w = w'$. For every choice of $\{w,w'\} \in E(G)$ or $w = w' \in V$, the expected number of pairs $\{P_{uv}, P_{u'v'}\}$ with $w \in P_{uv}$ and $w' \in P_{u'v'}$ is easily seen to be bounded above by $4\,\mathrm{vcong}(G)^2$ (using linearity of expectation and independence). Thus, the total expected number of intersecting pairs of edges of K_V is at most $4(m+n)\,\mathrm{vcong}(G)^2 \leqslant 4(2m+1)\,\mathrm{vcong}(G)^2$.

At the same time, it is well known that $\mathrm{pcr}(K_V) = \Omega(n^4)$, *i.e.*, any drawing of K_V has $\Omega(n^4)$ intersecting pairs of edges (see, *e.g.*, [8, Theorem 3]). So $m\,\mathrm{vcong}(G)^2 = \Omega(n^4)$ and the lemma follows. □

Vertex congestion and separators. Let us define

$$\mathrm{vcong}^*(G) := \min\{\mathrm{vcong}(H) : H \text{ is an induced subgraph of } G \text{ on at least } \tfrac{2}{3}n \text{ vertices}\}.$$

Lemma 3. *Every graph G on n vertices has a vertex separator with $O((n^2 \log n)/\,\mathrm{vcong}^*(G))$ vertices.*

Proof. The proof goes in the following steps, all of them contained in [2] (see also [1], especially Section 5.2 there, for a similar use of [2]).

(1) Let $s\colon V \to [0,\infty)$ be an assignment of real weights to the vertices of G, let the weight of an edge $e = \{u,v\} \in E(G)$ be $(s(u)+s(v))/2$, and let d_s be the shortest-path pseudometric in G with these edge weights. By the duality of linear programming, it

[1] It is well known, and easy to check by a compactness argument, that min is attained.

is easy to derive (see [2, Section 4])

$$\frac{1}{\mathrm{vcong}(G)} = \min \left\{ \sum_{v \in V} s(v) : \sum_{\{u,v\} \in \binom{V}{2}} d_s(u,v) = 1 \right\}.$$

(2) Let s^* be a vertex weighting attaining the minimum in the last formula. By suitable use of a famous result of Bourgain (see [2, Theorem 3.1]), we get that there exists a function $f : V \to \mathbb{R}$ that is 1-Lipschitz with respect to s^*, *i.e.*, $|f(u) - f(v)| \leqslant d_{s^*}(u,v)$ for all $u, v \in V$, and such that

$$\sum_{\{u,v\} \in \binom{V}{2}} |f(u) - f(v)| = \Omega\left(\left(\sum_{\{u,v\} \in \binom{V}{2}} d_{s^*}(u,v) \right) / \log n \right) = \Omega(1/\log n).$$

(3) Let (A, B, S) be a partition of the vertex set of a graph G into three disjoint subsets with $A \neq \emptyset \neq B$ and no edges between A and B. Let the *sparsity* of (A, B, S) be

$$\frac{|S|}{|A \cup S| \cdot |B \cup S|}.$$

By [2, Lemma 3.7], given a function f as above for G, there exists a partition (A, B, S) of the vertex set with sparsity

$$O\left(\left(\sum_{v \in V} s^*(v) \right) \log n \right) = O((\log n)/\,\mathrm{vcong}(G)).$$

(4) A standard procedure, starting with G and repeatedly finding a sparse partition until the size of all components drops below $\frac{2}{3}n$ (see, *e.g.*, [2, Section 6]), then finds a separator of size $O((n^2 \log n)/\,\mathrm{vcong}^*(G))$ in G as claimed. □

Remark. Although Lemma 3 is tight for arbitrary graphs, a possible way towards proving the Fox–Pach conjecture, separators for string graphs of size $O(\sqrt{m})$, would be removing the $\log n$ factor in Lemma 3 under the assumption that G is a string graph. More concretely, the improvement might be achievable in item (2) of the proof above: indeed, if G is a planar graph or, more generally, belongs to a minor-closed class of graphs with a forbidden minor, then, in the setting of item (2), the 1-Lipschitz f can even be made to satisfy

$$\sum_{\{u,v\} \in \binom{V}{2}} |f(u) - f(v)| = \Omega(1)$$

([9]; see also [2, Theorem 3.2]). Thus, a similar improvement for string graphs is perhaps not out of reach.

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